PHYS 101 – General Physics I Midterm Exam

Duration: 90 minutes Saturday, 14 October 2023; 13:30

1. A point particle is moving on the *x*-axis with constant acceleration. At time $t = 1$s, it is observed to be at the origin (i.e., $x(1) = 0$) and not moving (*i.e.*, $v_x(1) = 0$). Later, at time $t = 2$s, its velocity is measured as $v_x(2) = 2\ \mathrm{m/s}$.

(a) (15 Pts.) Find the equation describing the motion of the object (i.e., find $x(t)$).

(b) (5 Pts.) What is the displacement of the object during the time interval $t = 0$ and $t = 2$s?

(c) (10 Pts.) What is the average speed of the object during the time interval $t = 0$ and $t = 2$s?

Solution:

(a) The positon and the velocity of a particle moving along the *x*-axis with constant acceleration is given by

$$x(t) = x_0 + v_{x0}t + \frac{1}{2}a_x t^2\,, \qquad v(t) = v_{x0} + a_x t\,.$$

Since $x(1) = 0$, $v_x(1) = 0$, and $v_x(2) = 2$ m/s, we have

$$x_0 + v_{x0} + \frac{1}{2}a_x = 0\,, \qquad v_{x0} + a_x = 0\,, \qquad v_{x0} + 2\,a_x = 2.$$

Solving these three equations, we find $x_0 = 1$ m, $v_{x0} = -2$ m/s, and $a_x = 2\ \mathrm{m/s^2}$. So, the equation describing the motion of the object is

$x(t) = 1 - 2\,t + t^2$ m.

(b) At time $t = 0$ the object is at $x(0) = 1$ m, and since $x(2) = 1$ m. Therefore, the displacement of the object during the time interval $t = 0$ and $t = 2$s is $x(2) - x(0) = 0$.

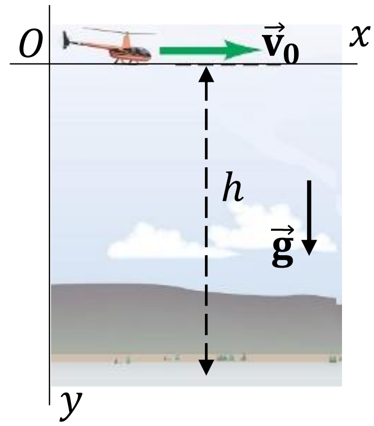
(c) Velocity of the object is $v_x = -2 + 2\,t$, which is zero at time $t = 1$ s when the object is at the origin. Since the direction of the velocity changes at time $t = 1$ s, the total distance Δd traveled by the object in two seconds is

$$\Delta d = |x(2) - x(1)| + |x(1) - x(0)| \quad \rightarrow \quad \Delta d = 2\,\mathrm{m}\,.$$

Therefore, the average speed s_{av} of the object during the time interval $t = 0$ and $t = 2$s is

$$s_{av} = \frac{\Delta d}{\Delta t} = 1\ \mathrm{m/s}\,.$$

2. At time $t = 0$, an object falls freely from a helicopter which is flying with constant horizontal velocity $\vec{\mathbf{v}}_\mathbf{0} = v_0\,\hat{\mathbf{i}}$, at an altitude h above a level road. The helicopter is at the origin of the coordinate system shown in the figure when the fall occurs. Because of a headwind, the falling object also has a constant horizontal acceleration $\vec{\mathbf{a}} = -a\,\hat{\mathbf{i}}$. Gravitational acceleration is $\vec{\mathbf{g}} = \mathrm{g}\,\hat{\mathbf{j}}$.



(a) (10 Pts.) What is the position vector of the object for $t > 0$ in the given coordinate frame?

(b) (10 Pts.) What is the horizontal displacement of the object when it hits the road?

(c) (10 Pts.) What are the *x* and *y* components of the velocity of the object at the instant it hits the road?

Solution:

(a) Initial conditions for the falling object are $x_0 = 0,\ y_0 = 0,\ v_{x0} = v_0,\ v_{y0} = 0$. Therefore

$$\vec{\mathbf{r}}(t) = \left(v_0 t - \frac{1}{2} a\, t^2\right)\hat{\mathbf{i}} + \left(\frac{1}{2} g\, t^2\right)\hat{\mathbf{j}}\,.$$

(b) When the object hits the road at time t_h, we have $y(t_h) = h$, meaning that

$$\frac{1}{2} g\, t_h^2 = h \quad \rightarrow \quad t_h = \sqrt{\frac{2h}{g}}\,.$$

Horizontal displacement of the object during this time is

$$x(t_h) = v_0 \sqrt{\frac{2h}{g}} - \frac{a\,h}{g}\,.$$

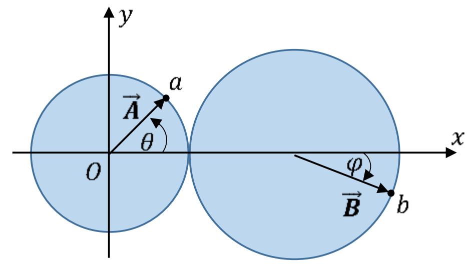
(c) The velocity of the object is

$$\vec{\mathbf{v}} = (v_0 - a\,t)\hat{\mathbf{i}} + (g\,t)\,\hat{\mathbf{j}}\,.$$

Therefore, when it hits the road

$$v_x(t_h) = v_0 - a\sqrt{\frac{2h}{g}}\,, \qquad v_y(t_h) = g\sqrt{\frac{2h}{g}} = \sqrt{2gh}\,.$$

3. Two gears with radii $r_1 = 1$ m and $r_2 = \sqrt{2}$ m are in non-sliding contact. One is turning clockwise, while the other is turning counter clockwise, and both angular velocities are constant. The vector $\vec{\boldsymbol{A}}(t)$ starts at the center of the first gear and points to the point a on the edge of the first gear. Similarly, the vector $\vec{\boldsymbol{B}}(t)$ starts from the center of the second gear and poits to the point b on the edge of the second gear. At time $t = 0$ both vectors are in the $+\hat{\mathbf{i}}$ direction. A point on the edge of the first gear moves with linear speed $v_1 = \pi$ m/s.



(a) (7 Pts.) Write $\vec{\boldsymbol{A}}(t)$ using the coordinate frame shown in the figure.

(b) (7 Pts.) What is the acceleration (vector) of point a as a function of time?

(c) (7 Pts.) Write $\vec{\boldsymbol{B}}(2)$ (at time $t = 2$ s) using the coordinate frame shown in the figure.

(d) (7 Pts.) What is the maximum value of the relative speed of point b with respect to point a, throughout the motion?

(e) (7 Pts.) What is the maximum value for the magnitude of the vector $\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}$ throughout the motion?

Solution:

(a) The motion of the point a is uniform circular motion. Therefore

$$\vec{\boldsymbol{A}} = (\cos\theta)\hat{\mathbf{i}} + (\sin\theta)\hat{\mathbf{j}} \text{ m.}$$

Since the linear speed of the point a is π m/s, $v = r\,\omega \rightarrow \omega = \pi \text{ s}^{-1}$ and $\theta = \omega\, t = \pi t$. Therefore

$$\vec{\boldsymbol{A}}(t) = (\cos\pi t)\hat{\mathbf{i}} + (\sin\pi t)\hat{\mathbf{j}} \text{ m.}$$

(b)

$$\vec{\mathbf{a}}_a = \frac{d^2\vec{\boldsymbol{A}}}{dt^2} \rightarrow \vec{\mathbf{a}}_a = -\pi^2(\cos\pi t)\hat{\mathbf{i}} - \pi^2(\sin\pi t)\hat{\mathbf{j}} \text{ m/s}^2.$$

(c) Magnitude of the vector $\vec{\boldsymbol{B}}$ is $\sqrt{2}$. Since it is rotating in the opposite direction, we have

$$\vec{\boldsymbol{B}}(t) = \sqrt{2}(\cos\varphi)\hat{\mathbf{i}} - \sqrt{2}(\sin\varphi)\hat{\mathbf{j}} \text{ m.}$$

The two gears are in non-sliding contace means $r_1\theta = r_2\varphi \rightarrow \varphi = \pi t/\sqrt{2}$. Therefore

$$\vec{\boldsymbol{B}}(t) = \sqrt{2}\left[\cos\left(\frac{\pi t}{\sqrt{2}}\right)\right]\hat{\mathbf{i}} - \sqrt{2}\left[\sin\left(\frac{\pi t}{\sqrt{2}}\right)\right]\hat{\mathbf{j}} \text{ m} \quad \rightarrow \quad \vec{\boldsymbol{B}}(2) = \sqrt{2}\big[\cos\big(\sqrt{2}\,\pi\big)\big]\hat{\mathbf{i}} - \sqrt{2}\big[\sin\big(\sqrt{2}\,\pi\big)\big]\hat{\mathbf{j}} \text{ m.}$$

(d) At certain times during the motion the linear speed of point b will be in the opposite direction of the linear speed of point a. Since the linear speeds of both points are π, the maximum value of the relative speed of point b with respect to point a is 2π m/s.

(d)

$$\left|\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}\right| = \left|\vec{\boldsymbol{A}}\right|\left|\vec{\boldsymbol{B}}\right|\left|\sin(\theta + \varphi)\right| .$$

Since the maximum value of $|\sin(\theta + \varphi)|$ is 1, $\left|\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}\right|_{\text{max}} = \left|\vec{\boldsymbol{A}}\right|\left|\vec{\boldsymbol{B}}\right| = \sqrt{2} \text{ m}^2$.